HW6 - 4050 March 2024

1. Show the inequalities relating to lower/upper Riemann/Lebesgue integrals for any bounded function f on $[a,b]$:

$$(R)\underline{\int_a^b} f \le (L)\underline{\int_a^b} \le (L)\overline{\int_a^b} f \le (R)\overline{\int_b^a} f .$$

Show further that if $f \in R[a,b]$ then

$$(L)\underline{\int_a^b} f = (L)\overline{\int_a^b} f = (R)\int_a^b f .$$

2. Let $f \in \mathcal{BF}(E)$ with $m(E) < +\infty$ be s.t. $\underline{\int_E} f = \overline{\int_E} f$. Show that $\exists\ \varphi_n, \psi_n \in \mathcal{S}(E)$ with $\varphi_n \le f \le \psi_n$ on E such that

$$\underline{\int_E} f - \frac{1}{n} < \int_E \varphi_n \ \ \& \ \ \int_E \psi_n < \overline{\int_E} f + \frac{1}{n}$$

Setting $\overline{\varphi} := \bigvee_{n=1}^{\infty} \varphi_n$ & $\underline{\psi} := \bigwedge_{n=1}^{\infty} \psi_n$, show that $\overline{\varphi} = f = \underline{\psi}$ a.e on E and that f is measurable.

3. Assuming the corresponding property for $\mathcal{S}_0$ and $\mathcal{B}\mathcal{M}\mathcal{F}_0(E)$, show that

(a) $\int_F f = \int_E (f\chi_F)$ $\forall f \in \mathcal{M}\mathcal{F}^+(E)$, $\forall \mathcal{M} \ni F \subseteq E$;

(b) Is (a) true for $\mathcal{L}(E) := \{ f \in \mathcal{M}\mathcal{F}(E) : \int_E |f| < +\infty \}$?

4. Assuming the additivity and monotonicity of $f \mapsto \int_E f$ on $\mathcal{M}\mathcal{F}^+(E)$ show the corresponding property for $\mathcal{L}(E)$ and:

(a) $\int_E \left(\sum_{n=1}^{\infty} u_n \right) = \sum_{n=1}^{\infty} \int_E u_n$ $\forall u_n \in \mathcal{M}\mathcal{F}^+(E)$ $(n \in \mathbb{N})$.

(b) $\int_{\bigcup_{n=1}^{\infty} E_n} f = \sum_{n=1}^{\infty} \int_{E_n} f$, $\forall$ disjoint $\{E_n : n \in \mathbb{N}\} \subseteq \mathcal{M}$, $E_n \subseteq E$ $\forall n$.

(c) Let $f \in \mathcal{M}\mathcal{F}^+(E)$. Show that

$\int_E f = 0 \Rightarrow f = 0$ a.e on E

$\int_E f < +\infty \Rightarrow f(x) < +\infty$ a.e. x in E.

(Hint. Set $\Delta_0 := \{ x \in E : f(x) = 0 \}$, $\Delta_\delta := \{ x \in E : f(x) \geq \delta \}$ $(\delta > 0)$

$\Delta^{\infty} := \{ x \in E : f(x) = +\infty \} \subseteq \Delta^{(n)} := \{ x \in E : f(x) \geq n \}$, $\forall n \in \mathbb{N}$.

(d). Let $f \in \mathcal{L}(E)$ be s.t. $\int_F f = 0$ $\forall \mathcal{M} \ni F \subseteq E$.

Show that $f = 0$ a.e. on E, via considering

$F_1 := \{ x \in E : f(x) < 0 \}$, $F_2 := \{ x \in E : f(x) > 0 \}$.

(e) Let $f \in \mathcal{L}(E)$ be s.t. $\int_F f = 0 \ \forall$ closed $F \subseteq E$.

Show that $f = 0$ a.e. on E (Hint: inner regularity of E).

5. Let $f \in \mathcal{L}[a+c, b+c]$, $c \in \mathbb{R}$. Show that

$$\int_a^b f(x+c)\,dx = \int_{a+c}^{b+c} f(y)\,dy. \quad (\text{"change of variables" formulas})$$

regarding the Lebesgue integral of f on $[a+c, b+c]$ and that of the "translate"

$$x \mapsto f(x+c) \text{ on } [a,b]$$

(Hint: true for simple functions)

6. Show that

$$A \mapsto \int_A f$$

is ABC (absolutely continuous) in the sense that $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $\left|\int_A f\right| < \varepsilon$ whenever $m(A) < \delta$ and $A \subseteq E$ in each of the following cases:

(a) $m(E) \leq +\infty$, $f \in \mathcal{BMF}(E)$: $\exists M \in (0,\infty)$ s.t. $|f| \leq M$ on E

(b) $f \in \mathcal{MF}^+(E)$, $\int_E f < +\infty$ (Hint: take $\varphi_n \in \mathcal{S}_0^+(E)$, $\varphi_n \uparrow f$. Apply MCT & (a))

(c) $f \in \mathcal{L}(E)$.

7. Let $f \in \mathcal{L}[a,b]$ and

$$F(x) = \int_a^x f \quad \forall\, x \in [a,b].$$

Show that F is ABC in the sense that

$\forall\, \varepsilon > 0\ \exists\, \delta > 0$ s.t.

$$\sum_{i=1}^{n} |F(x_i'') - F(x_i')| < \varepsilon$$

whenever $\{(x_i', x_i'') : i = 1, 2, \cdots, n\}$ is a disjoint open intervals $\left(\subseteq (a,b) \right)$ of total length $< \delta$.

Can n be replaced by ∞?